5-2 Verifying Trigonometric Identities

Verify each identity.

1. $(\sec^2 \theta - 1)\cos^2 \theta = \sin^2 \theta$

SOLUTION:

$$
\begin{aligned}
&(\sec^2\theta - 1)\cos^2\theta \\
&= (\tan^2\theta)\cos^2\theta && \text{Pythagorean Identity} \\
&= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta && \text{Quotient Identity} \\
&= \sin^2\theta && \text{Multiply and divide out common factor.}
\end{aligned}
$$

3. $\sin\theta - \sin\theta\cos^2\theta = \sin^3\theta$

SOLUTION:

$$
\begin{aligned}
&\sin\theta - \sin\theta\cos^2\theta \\
&= \sin\theta(1 - \cos^2\theta) && \text{Factor.} \\
&= \sin\theta\sin^2\theta && \text{Pythagorean Identity} \\
&= \sin^3\theta && \text{Multiply.}
\end{aligned}
$$

5. $\cot^2\theta\csc^2\theta - \cot^2\theta = \cot^4\theta$

SOLUTION:

$$
\begin{aligned}
&\cot^2\theta\csc^2\theta - \cot^2\theta \\
&= \cot^2\theta(\csc^2\theta - 1) && \text{Factor.} \\
&= \cot^2\theta\cot^2\theta && \text{Pythagorean Identity} \\
&= \cot^4\theta && \text{Multiply and add exponents.}
\end{aligned}
$$

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7. $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$

SOLUTION:

$$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$ Reciprocal Identity

$$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$ Common denominator

$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta}$$ Write with a com denom

$$= \frac{\cos^2\theta}{\sin\theta\cos\theta}$$ Pythagoren Identity

$$= \frac{\cos\theta}{\sin\theta}$$ Divide out $\cos\theta$.

$$= \cot\theta$$ Quotient Identity

9. $\frac{\cos\theta}{1+\sin\theta} + \tan\theta = \sec\theta$

SOLUTION:

$$\frac{\cos\theta}{1+\sin\theta} + \tan\theta$$

$$= \frac{\cos\theta}{1+\sin\theta} + \frac{\sin\theta}{\cos\theta}$$ Quotient Identity

$$= \frac{\cos\theta}{\cos\theta}\cdot\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{1+\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}$$ Rewrite 1 with com denom

$$= \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)} + \frac{\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$$ Multiply

$$= \frac{\cos^2\theta+\sin\theta+\sin^2\theta}{\cos\theta(1+\sin\theta)}$$ Write as a single fraction

$$= \frac{1+\sin\theta}{\cos\theta(1+\sin\theta)}$$ Pythagorean Identity

$$= \frac{1}{\cos\theta}$$ Divide out $(1+\sin\theta)$

$$= \sec\theta$$ Reciprocal Identity

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11. $\dfrac{1}{1-\tan^2\theta}+\dfrac{1}{1-\cot^2\theta}=1$

SOLUTION:

$$\frac{1}{1-\tan^2\theta}+\frac{1}{1-\cot^2\theta}$$

$$=\frac{1}{1-\frac{\sin^2\theta}{\cos^2\theta}}+\frac{1}{1-\frac{\cos^2\theta}{\sin^2\theta}}$$ Quotient Identity

$$=\frac{1}{\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}}+\frac{1}{\frac{\sin^2\theta}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}}$$ Rewrite 1 using com denom

$$=\frac{1}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}+\frac{1}{\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta}}$$ Write denom as single fractions

$$=\frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta-\cos^2\theta}$$ Simplify fractions

$$=\frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}+\frac{-\sin^2\theta}{\cos^2\theta-\sin^2\theta}$$ Common denominator

$$=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta-\sin^2\theta}$$ Write as a single fraction

$=1$ Divide out $\left(\cos^2\theta-\sin^2\theta\right)$

13. $(\csc\ \theta\ -\cot\ \theta)(\csc\ \theta\ +\cot\ \theta)=1$

SOLUTION:

$$(\csc\theta-\cot\theta)(\csc\theta+\cot\theta)$$

$=\csc^2\theta-\cot^2\theta$ Multiply.

$=1$ Pythagorean Identity

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15. $\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=2\sec^2\theta$

SOLUTION:

$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}$$

$$=\frac{1+\sin\theta}{1+\sin\theta}\cdot\frac{1}{1-\sin\theta}+\frac{1-\sin\theta}{1-\sin\theta}\cdot\frac{1}{1+\sin\theta}$$ Common denominator

$$=\frac{1+\sin\theta}{1-\sin^2\theta}+\frac{1-\sin\theta}{1-\sin^2\theta}$$ Multiply

$$=\frac{2}{1-\sin^2\theta}$$ Write as a single fraction

$$=\frac{2}{\cos^2\theta}$$ Pythagorean Identity

$$=2\sec^2\theta$$ Reciprocal Identity

17. $\csc^4\theta-\cot^4\theta=2\cot^2\theta+1$

SOLUTION:

$$\csc^4\theta-\cot^4\theta$$

$$=(\csc^2\theta-\cot^2\theta)(\csc^2\theta+\cot^2\theta)$$ Factor

$$=[\csc^2\theta-(\csc^2\theta-1)][\csc^2\theta+(\csc^2\theta-1)]$$ Pythagorean Identity

$$=[\csc^2\theta-\csc^2\theta+1)][\csc^2\theta+\csc^2\theta-1]$$ Multiply

$$=[1][2\csc^2\theta-1]$$ Add

$$=2\csc^2\theta-1$$ Multiply

$$=2(\cot^2\theta+1)-1$$ Pythagorean Identity

$$=2\cot^2\theta+2-1$$ Multiply

$$=2\cot^2\theta+1$$ Add

5-2 Verifying Trigonometric Identities

Verify each identity.

21. $\sin^2\theta \tan^2\theta = \tan^2\theta - \sin^2\theta$

SOLUTION:

$$\tan^2\theta - \sin^2\theta$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \qquad \text{Quotient Identity}$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \cdot 1 \qquad \text{Multiply by 1}$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \left(\sin^2\theta\right)\left(\frac{\cos^2\theta}{\cos^2\theta}\right) \qquad \text{Rewrite 1 with com denom}$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta} \qquad \text{Multiply}$$

$$= \frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta} \qquad \text{Write as a single fraction}$$

$$= \frac{\sin^2\theta(1 - \cos^2\theta)}{\cos^2\theta} \qquad \text{Factor the numerator}$$

$$= \frac{\sin^2\theta\sin^2\theta}{\cos^2\theta} \qquad \text{Pythagorean Identity}$$

$$= \sin^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right) \qquad \text{Factor}$$

$$= \sin^2\theta\tan^2\theta \qquad \text{Quotient Identity}$$

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23. $\frac{1+\csc\theta}{\sec\theta} = \cos\theta + \cot\theta$

SOLUTION:

$$\frac{1+\csc\theta}{\sec\theta}$$

$$= \frac{1+\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}}$$ Reciprocal Identity

$$= \frac{\frac{\sin\theta}{\sin\theta}+\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}}$$ Rewrite 1 with com denom

$$= \frac{\frac{\sin\theta+1}{\sin\theta}}{\frac{1}{\cos\theta}}$$ Write the numerator as a single fraction

$$= \frac{\sin\theta+1}{\sin\theta}\cdot\frac{\cos\theta}{1}$$ Multiply by the reciprocal

$$= \frac{\sin\theta\cos\theta+\cos\theta}{\sin\theta}$$ Write as a single fraction

$$= \frac{\sin\theta\cos\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}$$ Write as two fractions

$$= \cos\theta+\frac{\cos\theta}{\sin\theta}$$ Divide out $\sin\theta$

$$= \cos\theta+\cot\theta$$ Quotient Identity

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25. $\dfrac{1+\tan^2\theta}{1-\tan^2\theta} = \dfrac{1}{2\cos^2\theta - 1}$

SOLUTION:

$$\frac{1+\tan^2\theta}{1-\tan^2\theta}$$

$$= \frac{1+\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}$$ Quotient Identity

$$= \frac{\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}}$$ Rewrite 1 with com denom

$$= \frac{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$$ Write numerator and denominator with a common denominator

$$= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}$$ Multiply by the reciprocal

$$= \frac{\cos^2\theta\left(\cos^2\theta+\sin^2\theta\right)}{\cos^2\theta\left(\cos^2\theta-\sin^2\theta\right)}$$ Multiply.

$$= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}$$ Divide out $\cos^2\theta$.

$$= \frac{1}{\cos^2\theta-\sin^2\theta}$$ Pythagorean Identity

$$= \frac{1}{\cos^2\theta-(1-\cos^2\theta)}$$ Pythagorean Identity

$$= \frac{1}{\cos^2\theta-1+\cos^2\theta}$$ Distributive Property

$$= \frac{1}{2\cos^2\theta-1}$$ Simplify the denominator.