5-2 Verifying Trigonometric Identities

Verify each identity.

27. sec θ – cos θ = tan θ sin θ

SOLUTION:

$\sec\theta - \cos\theta$	
$= \dfrac{1}{\cos\theta} - \cos\theta$	Reciprocal Identity
$= \dfrac{1}{\cos\theta} - \dfrac{\cos^2\theta}{\cos\theta}$	Rewrite $\cos\theta$ using the common denominator.
$= \dfrac{1-\cos^2\theta}{\cos\theta}$	Add fractions.
$= \dfrac{\sin^2\theta}{\cos\theta}$	Pythagorean Identity
$= \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{\sin\theta}{1}$	Rewrite as a procuct of two fractions.
$= \tan\theta\sin\theta$	Quotient Identity

29. $\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \frac{1-\cos\theta}{1+\cos\theta}$

SOLUTION:

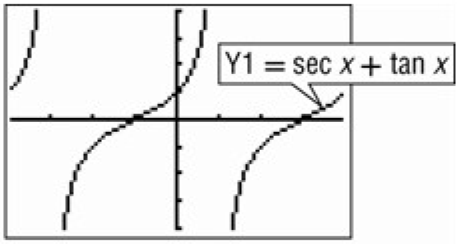
$\left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\right)^2 = \left(\dfrac{1-\cos\theta}{\sin\theta}\right)^2$	Add fractions
$= \dfrac{(1-\cos\theta)^2}{\sin^2\theta}$	Power of a Quotient
$= \dfrac{(1-\cos\theta)^2}{1-\cos^2\theta}$	Pythagorean Identity
$= \dfrac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$	Factor.
$= \dfrac{1-\cos\theta}{1+\cos\theta}$	Divide out common factor.

GRAPHING CALCULATOR Test whether each equation is an identity by graphing. If it appears to be an identity, verify it. If not, find an *x*-value for which both sides are defined but not equal.

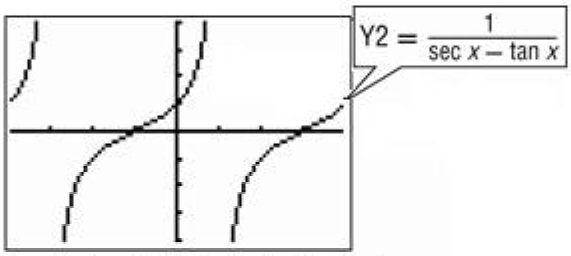
35. $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$

SOLUTION:

Graph $Y1 = \sec x + \tan x$ and then graph $Y2 = \dfrac{1}{\sec x - \tan x}$.



[−2π, 2π] scl: $\frac{\pi}{2}$ by [−4, 4] scl: 1



[−2π, 2π] scl: $\frac{\pi}{2}$ by [−4, 4] scl: 1

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

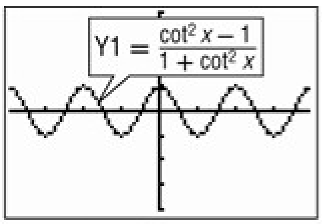
$\dfrac{1}{\sec x - \tan x} = \dfrac{1}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}}$	Reciprocal and Quotient Identities
$= \dfrac{1}{\frac{1 - \sin x}{\cos x}}$	Subtract fractions in the denominator.
$= \dfrac{\cos x}{1 - \sin x}$	$1 \div \dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$
$= \dfrac{\cos x}{1 - \sin x} \cdot \dfrac{1 + \sin x}{1 + \sin x}$	Multiply numerator and denominator by the conjugate of the denominator.
$= \dfrac{\cos x + \sin x \cos x}{1 - \sin^2 x}$	Multiply.
$= \dfrac{\cos x + \sin x \cos x}{\cos^2 x}$	Pythagorean Identity
$= \dfrac{\cos x}{\cos^2 x} + \dfrac{\sin x \cos x}{\cos^2 x}$	Write as a sum of two fractions.
$= \dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}$	Divide out the common factor $\cos x$.
$= \sec x + \tan x$	Reciprocal and Quotient Identities.

5-2 Verifying Trigonometric Identities

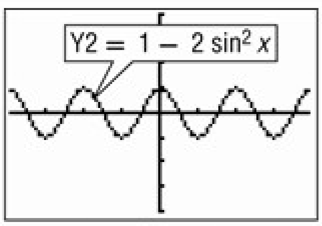
37. $\dfrac{\cot^2 x - 1}{1 + \cot^2 x} = 1 - 2\sin^2 x$

SOLUTION:

Graph **Y1** = $\dfrac{\cot^2 x - 1}{1 + \cot^2 x}$ and then graph **Y2** = $1 - 2\sin^2 x$.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

$$\frac{\cot^2 x - 1}{1 + \cot^2 x} = \frac{\cot^2 x - 1}{\csc^2 x}$$ Pythagorean Identity

$$= \frac{\frac{\cos^2 x}{\sin^2 x} - 1}{\frac{1}{\sin^2 x}}$$ Quotient and Reciprocal Identities

$$= \left(\frac{\cos^2 x}{\sin^2 x} - 1\right)\sin^2 x$$ $1 \div \frac{1}{\sin^2 x} = \sin^2 x$

$$= \cos^2 x - \sin^2 x$$ Multiply and divide out common factor.

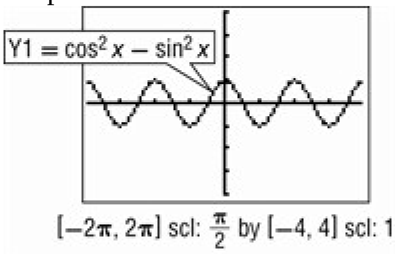
$$= (1 - \sin^2 x) - \sin^2 x$$ Pythagorean Identity

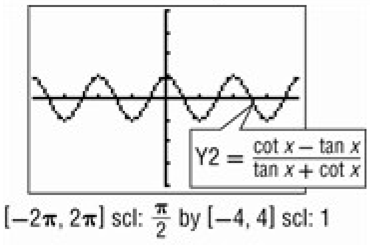
$$= 1 - 2\sin^2 x \ \checkmark$$ Simplify.

39. $\cos^2 x - \sin^2 x = \dfrac{\cot x - \tan x}{\tan x + \cot x}$

SOLUTION:

Graph **Y1** = $\cos^2 x - \sin^2 x$ and then graph **Y2** = $\dfrac{\cot x - \tan x}{\tan x + \cot x}$.





The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

$$\frac{\cot x - \tan x}{\tan x + \cot x}$$

$$= \frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}$$ Quotient Identities

$$= \frac{\frac{\cos^2 x}{\sin x \cos x} - \frac{\sin^2 x}{\sin x \cos x}}{\frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}}$$ Multiply to get common denominators

$$= \frac{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$$ Add fractions

$$= \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x}$$ Multiply numerator and denominator by $(\sin x \cdot \cos x)$

$$= \cos^2 x - \sin^2 x$$ Pythagorean Identity

Verify each identity.

45. $-2\cos^2\theta = \sin^4\theta - \cos^4\theta - 1$

SOLUTION:

Start with the right side of the identity.

$\sin^4\theta - \cos^4\theta - 1$	
$= (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) - 1$	Factor $(\sin^4\theta - \cos^4\theta)$.
$= (1)(1 - \cos^2\theta - \cos^2\theta) - 1$	Pythagorean Identities
$= 1 - \cos^2\theta - \cos^2\theta - 1$	Multiply.
$= -2\cos^2\theta$ ✓	Add.

47. $3\sec^2\theta\tan^2\theta + 1 = \sec^6\theta - \tan^6\theta$

SOLUTION:

Start with the right side of the identity.

$$\sec^6\theta - \tan^6\theta$$
$$= (\sec^3\theta - \tan^3\theta)(\sec^3\theta + \tan^3\theta)$$
$$= (\sec\theta - \tan\theta)(\sec^2\theta + \sec\theta\tan\theta + \tan^2\theta)(\sec\theta + \tan\theta)(\sec^2\theta - \sec\theta\tan\theta + \tan^2\theta)$$
$$= (\sec^2\theta - \tan^2\theta)[(1 + \tan^2\theta) + \sec\theta\tan\theta + \tan^2\theta]\cdot[(1 + \tan^2\theta) - \sec\theta\tan\theta + \tan^2\theta]$$
$$= (1 + 2\tan^2\theta + \sec\theta\tan\theta)(1 + 2\tan^2\theta - \sec\theta\tan\theta)$$
$$= (1 + 2\tan^2\theta)^2 - (\sec\theta\tan\theta)^2$$
$$= 1 + 4\tan^2\theta + 4\tan^4\theta - \sec^2\theta\tan^2\theta$$
$$= 1 + \tan^2\theta(4 + 4\tan^2\theta - \sec^2\theta)$$
$$= 1 + \tan^2\theta[4 + 4(\sec^2\theta - 1) - \sec^2\theta]$$
$$= 1 + \tan^2\theta(4 + 4\sec^2\theta - 4 - \sec^2\theta)$$
$$= 1 + \tan^2\theta(3\sec^2\theta)$$
$$= 1 + 3\tan^2\theta\sec^2\theta$$
$$= 3\sec^2\theta\tan^2\theta + 1$$

49. $\sec^2 x\csc^2 x = \sec^2 x + \csc^2 x$

SOLUTION:

Start with the left side of the identity.

$\sec^2 x\csc^2 x$	
$= (\tan^2 x + 1)\csc^2 x$	Pythagorean Identity
$= \left(\frac{\sin^2 x}{\cos^2 x} + 1\right)\left(\frac{1}{\sin^2 x}\right)$	Quotient and Reciprocal Identities
$= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$	Multiply.
$= \sec^2 x + \csc^2 x$ ✓	Reciprocal Identities