5-3 Solving Trigonometric Equations

Solve each equation for all values of *x*.

1. $5 \sin x + 2 = \sin x$

SOLUTION:

$$\begin{aligned} 5 \sin x + 2 &= \sin x \\ 4 \sin x + 2 &= 0 \\ 2(2 \sin x + 1) &= 0 \\ 2 \sin x &= -1 \\ \sin x &= -\frac{1}{2} \end{aligned}$$

The period of sine is 2π, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions on this interval are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of 2π. Therefore, the general form of the solutions is $\frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi, n \in \mathbb{Z}$.

3. $2 = 4 \cos^2 x + 1$

SOLUTION:

$$\begin{aligned} 2 &= 4\cos^2 x + 1 \\ 1 &= 4\cos^2 x \\ \frac{1}{4} &= \cos^2 x \\ \pm\sqrt{\frac{1}{4}} &= \sqrt{\cos^2 x} \\ \cos x &= \pm\frac{1}{2} \end{aligned}$$

The period of cosine is 2π, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions on this interval are $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$. Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of 2π.

Therefore, the general form of the solutions is $\frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in \mathbb{Z}$.

5-3 Solving Trigonometric Equations

5. $9 + \cot^2 x = 12$

SOLUTION:

$$\begin{aligned} 9+\cot^2 x &= 12 \\ \cot^2 x &= 3 \\ \sqrt{\cot^2 x} &= \pm\sqrt{3} \\ \cot x &= \pm\sqrt{3} \end{aligned}$$

The period of cotangent is π, so you only need to find solutions on the interval $[0, \pi)$. The solutions on this interval are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of π. Therefore, the general form of the solutions is $\frac{\pi}{6} + n\pi,\ \frac{5\pi}{6} + n\pi,\ n \in \mathbb{Z}$.

7. $3 \csc x = 2 \csc x + \sqrt{2}$

SOLUTION:

$$\begin{aligned} 3\csc x &= 2\csc x + \sqrt{2} \\ \csc x &= \sqrt{2} \end{aligned}$$

The period of cosecant is 2π, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions on this interval are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of 2π. Therefore, the general form of the solutions is $\frac{\pi}{4} + 2n\pi,\ \frac{3\pi}{4} + 2n\pi,\ n \in \mathbb{Z}$.

9. $6 \tan^2 x - 2 = 4$

SOLUTION:

$$\begin{aligned} 6\tan^2 x - 2 &= 4 \\ 6\tan^2 x &= 6 \\ \tan^2 x &= 1 \\ \sqrt{\tan^2 x} &= \pm\sqrt{1} \\ \tan x &= \pm 1 \end{aligned}$$

The period of tangent is π, so you only need to find solutions on the interval $[0, \pi)$. The solutions on this interval are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of π. Therefore, the general form of the solutions is $\frac{\pi}{4} + n\pi,\ \frac{3\pi}{4} + n\pi,\ n \in \mathbb{Z}$.

5-3 Solving Trigonometric Equations

11. $7 \cot x - \sqrt{3} = 4 \cot x$

SOLUTION:

$$\begin{aligned} 7\cot x - \sqrt{3} &= 4\cot x \\ 7\cot x &= 4\cot x + \sqrt{3} \\ 3\cot x &= \sqrt{3} \\ \cot x &= \frac{\sqrt{3}}{3} \end{aligned}$$

The period of cotangent is π, so you only need to find solutions on the interval $[0, \pi)$. The only solution on this interval is $\frac{\pi}{3}$. Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of π. Therefore, the general form of the solutions is $\frac{\pi}{3} + n\pi$, $n \in \mathbb{Z}$.

Find all solutions of each equation on [0, 2π).

13. $\sin^4 x + 2 \sin^2 x - 3 = 0$

SOLUTION:

$$\begin{aligned} \sin^4 x + 2\sin^2 x - 3 &= 0 \\ (\sin^2 x)^2 + 2\sin^2 x - 3 &= 0 \\ (\sin^2 x - 1)(\sin^2 x + 3) &= 0 \end{aligned}$$

$$\begin{aligned} \sin^2 x - 1 &= 0 \quad \text{or} \quad & \sin^2 x + 3 &= 0 \\ \sin^2 x &= 1 & \sin^2 x &= -3 \\ \sin x &= \pm\sqrt{1} & \sin x &= \pm\sqrt{-3} \\ \sin x &= \pm 1 \end{aligned}$$

On the interval [0, 2π), $\sin x = 1$ when $x = \frac{\pi}{2}$ and $\sin x = -1$ when $x = \frac{3\pi}{2}$. Since $\sqrt{-3}$ is not a real number, the equation $\sin x = \pm\sqrt{-3}$ yields no additional solutions.

15. $4 \cot x = \cot x \sin^2 x$

SOLUTION:

$$\begin{aligned} 4\cot x &= \cot x \sin^2 x \\ 4\cot x - \cot x \sin^2 x &= 0 \\ \cot x(4 - \sin^2 x) &= 0 \\ \cot x(2 - \sin x)(2 + \sin x) &= 0 \end{aligned}$$

$$\begin{aligned} \cot x = 0 \quad \text{or} \quad 2 - \sin x &= 0 \quad \text{or} \quad 2 + \sin x = 0 \\ -\sin x &= -2 \qquad\quad \sin x = -2 \\ \sin x &= 2 \end{aligned}$$

The equations $\sin x = 2$ and $\sin x = -2$ have no real solutions. On the interval [0, 2π), the equation $\cot x = 0$ has solutions $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

5-3 Solving Trigonometric Equations

17. $\cos^3 x + \cos^2 x - \cos x = 1$

SOLUTION:

$$\begin{aligned}
\cos^3 x + \cos^2 x - \cos x &= 1 \\
(\cos^3 x + \cos^2 x) - \cos x - 1 &= 0 \\
\cos^2 x(\cos x + 1) - (\cos x + 1) &= 0 \\
(\cos^2 x - 1)(\cos x + 1) &= 0 \\
(\cos x - 1)(\cos x + 1)^2 &= 0
\end{aligned}$$

$$\cos x - 1 = 0 \quad \text{or} \quad (\cos x + 1)^2 = 0$$

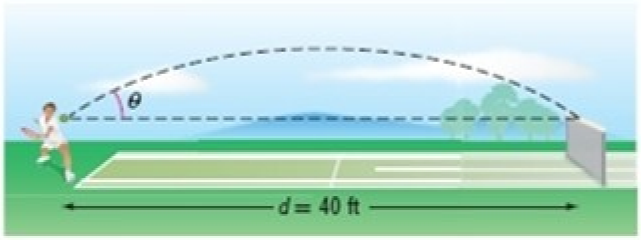
$$\cos x = 1 \qquad \cos x + 1 = 0$$

$$\cos x = -1$$

On the interval $[0, 2\pi)$, the equation $\cos x = 1$ has a solution of 0 and the equation $\cos x = -1$ has a solution of π.

5-3 Solving Trigonometric Equations

19. **TENNIS** A tennis ball leaves a racquet and heads toward a net 40 feet away. The height of the net is the same height as the initial height of the tennis ball.



a. If the ball is hit at 50 feet per second, neglecting air resistance, use $d = \frac{1}{32}{v_0}^2 \sin 2\theta$ to find the interval of possible angles of the ball needed to clear the net.
b. Find θ if the initial velocity remained the same but the distance to the net was 50 feet.

SOLUTION:

a.

$$d = \frac{1}{32}{v_0}^2 \sin 2\theta$$
$$40 = \frac{1}{32}(50)^2 \sin 2\theta$$
$$1280 = 2500 \sin 2\theta$$
$$\sin 2\theta = \frac{1280}{2500}$$
$$2\theta = \sin^{-1}\left(\frac{1280}{2500}\right)$$
$$2\theta = 30.8° \text{ or } 180° - 30.8°$$
$$2\theta = 30.8° \text{ or } 149.2°$$
$$\theta = 15.4° \text{ or } 74.6°$$

The interval is [15.4°, 74.6°].

b.

$$d = \frac{1}{32}{v_0}^2 \sin 2\theta$$
$$50 = \frac{1}{32}(50)^2 \sin 2\theta$$
$$1600 = 2500 \sin 2\theta$$
$$\sin 2\theta = \frac{1600}{2500}$$
$$2\theta = \sin^{-1}\left(\frac{1600}{2500}\right)$$
$$2\theta = 39.8° \text{ or } 180° - 39.8°$$
$$2\theta = 39.8° \text{ or } 140.2°$$
$$\theta = 19.9° \text{ or } 70.1°$$

If the distance to the net is 50 feet, then the angle would be 19.9° or 70.1°.

5-3 Solving Trigonometric Equations

Find all solutions of each equation on the interval [0, 2π).

21. $1 = \cot^2 x + \csc x$

SOLUTION:

$1 = \cot^2 x + \csc x$	Original equation.
$1 = \csc^2 x - 1 + \csc x$	Pythagorean Identity
$0 = \csc^2 x + \csc x - 2$	Subtract 1 from each side.
$0 = (\csc x - 1)(\csc x + 2)$	Factor.
$\csc x - 1 = 0$ or $\csc x + 2 = 0$	Zero Product Property
$\csc x = 1$ $\quad$ $\csc x = -2$	Solve for $\csc x$.
$x = \frac{\pi}{2}$ $\quad$ $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$	Solve for x on $[0, 2\pi)$.

CHECK $1 = \cot^2 \frac{\pi}{2} + \csc \frac{\pi}{2}$

$1 = 0^2 + 1$

$1 = 1$ √

$1 = \cot^2 \frac{7\pi}{6} + \csc \frac{7\pi}{6}$

$1 = \left(\sqrt{3}\right)^2 + (-2)$

$1 = 1$ √

$1 = \cot^2 \frac{11\pi}{6} + \csc \frac{11\pi}{6}$

$1 = \left(\sqrt{3}\right)^2 + (-2)$

$1 = 1$ √

Therefore, on the interval [0, 2π) the solutions are $\frac{\pi}{2}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.