4-3 Trigonometric Functions on the Unit Circle

Find the exact values of the five remaining trigonometric functions of θ.

33. $\tan\theta = 2$, where $\sin\theta > 0$ and $\cos\theta > 0$

SOLUTION:

To find the other function values, you must find the coordinates of a point on the terminal side of θ. You know that $\sin\theta$ and $\cos\theta$ are positive, so θ must lie in Quadrant I. This means that both x and y are positive.

Because $\tan\theta = \frac{y}{x}$ or $\frac{2}{1}$, use the point (1, 2) to find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5} \end{aligned}$$

Use $x = 1, y = 2$, and $r = \sqrt{5}$ to write the five remaining trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{5}} \text{ or } \frac{2\sqrt{5}}{5} \quad \csc\theta = \frac{r}{y} \text{ or } \frac{\sqrt{5}}{2}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{5}} \text{ or } \frac{\sqrt{5}}{5} \quad \sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{1} \text{ or } 1$$

$$\cot\theta = \frac{x}{y} \text{ or } \frac{1}{2}$$

35. $\sin\theta = -\frac{1}{5}$, where $\cos\theta > 0$

SOLUTION:

To find the other function values, you must find the coordinates of a point on the terminal side of θ. You know that $\sin\theta$ is negative and $\cos\theta$ is positive, so θ must lie in Quadrant IV. This means that x is positive and y is negative.

Because $\sin\theta = \frac{y}{r}$ or $-\frac{1}{5}$, use the point $(x, -1)$ and $r = 5$ to find x.

$$\begin{aligned} 5 &= \sqrt{x^2 + (-1)^2} \\ 5^2 &= \left(\sqrt{x^2 + 1}\right)^2 \\ 25 &= x^2 + 1 \\ x^2 &= 24 \\ x &= 2\sqrt{6} \end{aligned}$$

Use $x = 2\sqrt{6}$, $y = -1$, and $r = 5$ to write the five remaining trigonometric ratios.

$$\cos\theta = \frac{x}{r} \text{ or } \frac{2\sqrt{6}}{5} \qquad \csc\theta = \frac{r}{y} = \text{or } -5$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{2\sqrt{6}} \text{ or } = -\frac{\sqrt{6}}{12} \qquad \sec\theta = \frac{r}{x} = \frac{5}{2\sqrt{6}} \text{ or } \frac{5\sqrt{6}}{12}$$

$$\cot\theta = \frac{x}{y} \text{ or } -2\sqrt{6}$$

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37. sec $\theta = \sqrt{3}$, where sin $\theta < 0$ and cos $\theta > 0$

SOLUTION:

To find the other function values, you must find the coordinates of a point on the terminal side of θ. You know that sin θ is negative and cos θ is positive, so θ must lie in Quadrant IV. This means that x is positive and y is negative.

Because sec $\theta = \frac{r}{x}$ or $\frac{\sqrt{3}}{1}$, use the point $(1, y)$ and $r = \sqrt{3}$ to find y.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \left(\sqrt{3}\right)^2 &= \left(\sqrt{1^2 + y^2}\right)^2 \\ 3 &= 1 + y^2 \\ y^2 &= 2 \\ y &= -\sqrt{2} \end{aligned}$$

Use $x = 1, y = -\sqrt{2}$, and $r = \sqrt{3}$ to write the five remaining trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{2}}{\sqrt{3}} \text{ or } -\frac{\sqrt{6}}{3} \quad \csc\theta = \frac{r}{y} = \frac{\sqrt{3}}{-\sqrt{2}} \text{ or } -\frac{\sqrt{6}}{2}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \quad \tan = \frac{y}{x} = \frac{-\sqrt{2}}{1} \text{ or } -\sqrt{2}$$

$$\cot\theta = \frac{x}{y} = \frac{1}{-\sqrt{2}} \text{ or } -\frac{\sqrt{2}}{2}$$

39. tan $\theta = -1$, where sin $\theta < 0$

SOLUTION:

To find the other function values, you must find the coordinates of a point on the terminal side of θ. You know that sin θ is negative and cos θ is positive, so θ must lie in Quadrant IV. This means that x is positive and y is negative.

Because tan $\theta = \frac{y}{x}$ or $\frac{-1}{1}$, use the point $(1, -1)$ to find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ r &= \sqrt{(1)^2 + (-1)^2} \\ r &= \sqrt{2} \end{aligned}$$

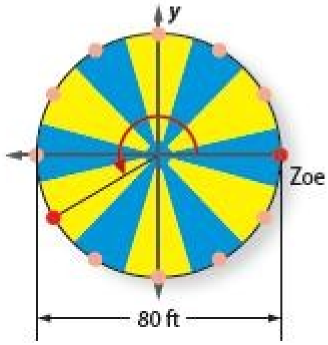
Use $x = 1, y = -1$, and $r = \sqrt{2}$ to write the five remaining trigonometric ratios.

$$\sin\theta = \frac{y}{r} = -\frac{1}{\sqrt{2}} \text{ or } -\frac{\sqrt{2}}{2} \qquad \csc\theta = \frac{r}{y} \text{ or } -\sqrt{2}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2} \quad \sec\theta = \frac{r}{x} \text{ or } \sqrt{2}$$

$$\cot\theta = \frac{x}{y} \text{ or } -1$$

41. **CAROUSEL** Zoe is on a carousel at the carnival. The diameter of the carousel is 80 feet. Find the position of her seat from the center of the carousel after a rotation of 210º.



SOLUTION:

Let the center of the carousel represent the origin on the coordinate plane and Zoe's position after the 210° rotation have coordinates (x, y). The definitions of sine and cosine can then be used to find the values of x and y. The value of r is 80 ÷ 2 or 40.

The seat rotates 210º, so the reference angle is 210º – 180º or 30º. Because the final position of the seat corresponds to Quadrant III, the sine and cosine of 210° are negative.

$$\begin{aligned} \cos\theta &= \frac{x}{r} \\ \cos 210° &= \frac{x}{40} \\ -\cos 30° &= \frac{x}{40} \\ -\frac{\sqrt{3}}{2} &= \frac{x}{40} \\ -\frac{40\sqrt{3}}{2} &= x \\ -20\sqrt{3} &= x \end{aligned}$$

$$\begin{aligned} \sin\theta &= \frac{y}{r} \\ \sin 210° &= \frac{y}{40} \\ -\sin 30° &= \frac{y}{40} \\ -\frac{1}{2} &= \frac{y}{40} \\ -\frac{40}{2} &= y \\ -20 &= y \end{aligned}$$

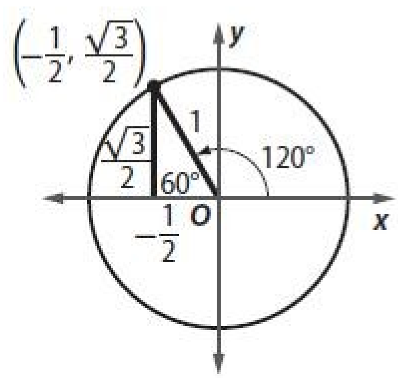
Therefore, the position of her seat relative to the center of the carousel is $\left(-20\sqrt{3}, -20\right)$ or (–34.6, –20).

Find the exact value of each expression. If undefined, write *undefined*.

43. sec 120°

SOLUTION:

120º corresponds to the point $(x, y) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.



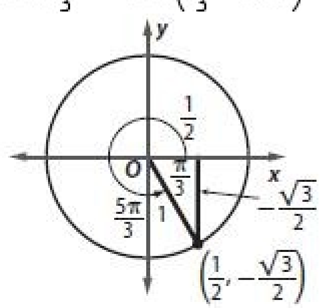
$$\sec t = \frac{1}{x}$$
$$\sec 120° = \frac{1}{-\frac{1}{2}} \text{ or } -2$$

45. $\cos \frac{11\pi}{3}$

SOLUTION:

Rewrite $\frac{11\pi}{3}$ as the sum of $\frac{5\pi}{3}$ and 2π

$$\cos\frac{11\pi}{3} = \cos\left(\frac{5\pi}{3} + 2\pi\right)$$



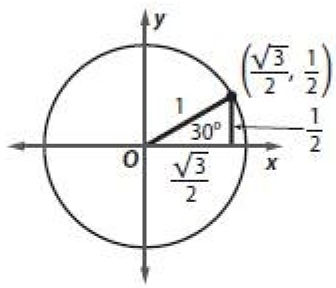
$$\cos\frac{11\pi}{3} = \cos\frac{5\pi}{3}$$
$$= \frac{1}{2}$$

47. csc 390°

SOLUTION:

Rewrite 390° as the sum of 30° and 360°.

$$\csc 390° = \csc(30° + 360°)$$



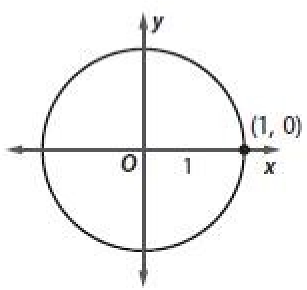
$$\begin{aligned}\csc 390° &= \csc 30° \\ &= \frac{1}{\sin 30} \\ &= \frac{1}{\frac{1}{2}} \text{ or } 2\end{aligned}$$

49. csc 5400°

SOLUTION:

Rewrite 5400 as the sum of 0° and 15×360°.

$$\csc 5400° = \csc(0° + 360(15)°)$$



$$\begin{aligned}\csc 5400° &= \csc 0° \\ &= \frac{1}{\sin 0°} \\ &= \frac{1}{0}\end{aligned}$$

Therefore, csc 5400° is undefined.

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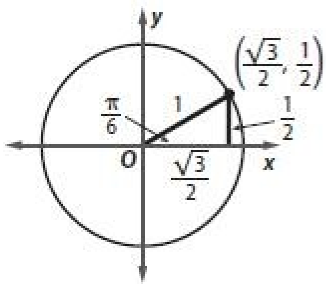
51. $\cot\left(-\frac{5\pi}{6}\right)$

SOLUTION:

Rewrite $-\frac{5\pi}{6}$ as a sum of $\frac{\pi}{6}$ and $-\pi$.

$\cot\left(-\frac{5\pi}{6}\right) = \cot\left(\frac{\pi}{6} + (-1)\pi\right)$

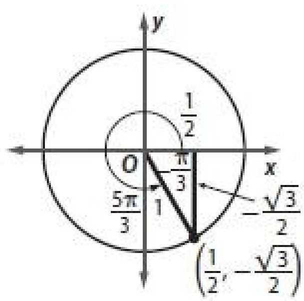


$$\begin{aligned}\cot\left(-\frac{5\pi}{6}\right) &= \frac{\pi}{6} \\ &= = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} \\ &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \text{ or } \sqrt{3}\end{aligned}$$

53. $\tan \frac{5\pi}{3}$

SOLUTION:

$\frac{5\pi}{3}$ corresponds to the point $(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the unit circle.



$$\tan t = \frac{y}{x}$$

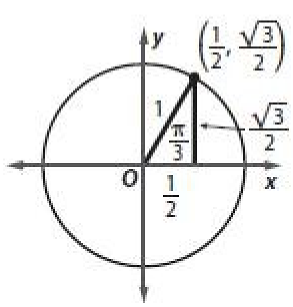
$$\tan\left(\frac{5\pi}{3}\right) = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \text{ or } -\sqrt{3}$$

55. $\sin\left(-\frac{5\pi}{3}\right)$

SOLUTION:

Rewrite $-\frac{5\pi}{3}$ as the sum of $\frac{\pi}{3}$ and –2 times 2π.

$$\sin\left(-\frac{5\pi}{3}\right) = \sin\left(\frac{\pi}{3} + (-2)\pi\right)$$



$$\sin\left(-\frac{5\pi}{3}\right) = \sin\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2}$$

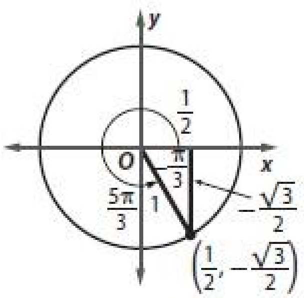
4-3 Trigonometric Functions on the Unit Circle

57. $\tan \frac{14\pi}{3}$

SOLUTION:

Rewrite $\frac{14\pi}{3}$ as the sum of $\frac{5\pi}{3}$ and 3 times π.

$\tan\frac{14\pi}{3} = \tan\left(\frac{5\pi}{3} + (3)\pi\right)$



$$\tan\frac{14\pi}{3} = \tan\frac{5\pi}{3}$$
$$= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \text{ or } -\sqrt{3}$$

Complete each trigonometric expression.

63. $\cos \frac{7\pi}{6} = \sin$ ___

SOLUTION:

$\frac{7\pi}{6}$ corresponds to the point $(x, y) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle. So, $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$.

On the unit circle, $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ and $\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$. Therefore, $\cos \frac{7\pi}{6} = \sin \frac{4\pi}{3}$ or $\cos \frac{7\pi}{6} = \sin \frac{5\pi}{3}$.

65. $\cos \frac{5\pi}{3} = \sin$ ___

SOLUTION:

$\frac{5\pi}{3}$ corresponds to the point $(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the unit circle. So, $\cos \frac{5\pi}{3} = \frac{1}{2}$.

On the unit circle, $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$. Therefore, $\cos \frac{5\pi}{3} = \sin \frac{\pi}{6}$ or $\cos \frac{5\pi}{3} = \sin \frac{5\pi}{6}$.

Use the given values to evaluate the trigonometric functions.

67. $\cos(-\theta) = \frac{8}{11}$; $\cos\theta =$?; $\sec\theta =$?

SOLUTION:

Because $\cos(-\theta) = \frac{8}{11}$ and $\cos(-\theta) = \cos\theta$, $\cos\theta = \frac{8}{11}$. So, $\sec\theta = \frac{1}{\cos\theta}$ or $\frac{11}{8}$.

69. $\sec\theta = \frac{13}{12}$; $\cos\theta =$?; $\cos(-\theta) =$?

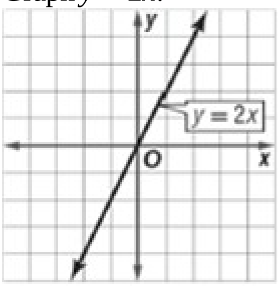
SOLUTION:

$$\sec\theta = \frac{13}{12}$$
$$\cos\theta = \frac{12}{13}$$
$$\cos(-\theta) = \frac{12}{13}$$

71. **GRAPHS** Suppose the terminal side of an angle θ in standard position coincides with the graph of $y = 2x$ in Quadrant III. Find the six trigonometric functions of θ.

SOLUTION:

Graph $y = 2x$.



One point that lies on the line in Quadrant III is (−2, −4). So, $x = -2$ and $y = -4$. Find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-2)^2 + (-4)^2} \\ &= \sqrt{20} \text{ or } 2\sqrt{5} \end{aligned}$$

Use $x = -2$, $y = -4$, and $r = 2\sqrt{5}$ to write the six trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{-4}{2\sqrt{5}} \text{ or } -\frac{2\sqrt{5}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{2\sqrt{5}} \text{ or } -\frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{-2} \text{ or } 2$$

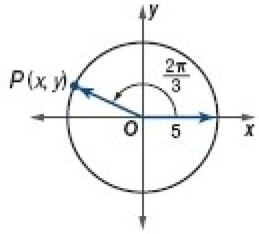
$$\csc\theta = \frac{r}{y} = \frac{2\sqrt{5}}{-4} \text{ or } -\frac{\sqrt{5}}{2}$$

$$\sec\theta = \frac{r}{x} = \frac{2\sqrt{5}}{-2} \text{ or } -\sqrt{5}$$

$$\cot\theta = \frac{x}{y} = \frac{-2}{-4} \text{ or } \frac{1}{2}$$

4-3 Trigonometric Functions on the Unit Circle

Find the coordinates of *P* for each circle with the given radius and angle measure.

73. 

SOLUTION:

Use the definitions of the cosine and sine functions to find the values of x and y. Because P is in Quadrant II, the cosine of $\frac{2\pi}{3}$ is negative and the sine of $\frac{2\pi}{3}$ is positive. The reference angle for $\frac{2\pi}{3}$ is $\frac{\pi}{3}$ and the radius r is 5.

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$\cos\frac{2\pi}{3} = \frac{x}{5} \qquad \sin\frac{2\pi}{3} = \frac{y}{5}$$

$$-\cos\frac{\pi}{3} = \frac{x}{5} \qquad \sin\frac{\pi}{3} = \frac{y}{5}$$

$$-\frac{1}{2} = \frac{x}{5} \qquad \frac{\sqrt{3}}{2} = \frac{y}{5}$$

$$-\frac{5}{2} = x \qquad \frac{5\sqrt{3}}{2} = y$$

So, the coordinates of P are $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.